Indian Statistical Institute
M.Math II Year
First Semester Examination, 2004-2005
Partial Differential Equations-I

Time: 3 hrs Date:25-11-04

Each problem carries five marks.

Answer all the questions. The problems are interrelated - you may use the results of problems $m_1, m_2, \ldots$ to solve problem n.

1. Show that every distribution with compact support belongs to $H^s(I\!R^n)$ for some $s \in I\!R$ whereas no distribution with finite support belongs to $H^{-\frac{n}{2}}(I\!R^n)$.

2. Show that every differential operator $P(X, D) = \sum\limits_{|\alpha| \leq k} a_\alpha(X)D^\alpha, x \in I\!R^n$ with C^∞ coefficients is hypoelliptic if and only if $n = 1$.

3. Show that $\partial_r^2 T\varphi(r) = T(\partial_r^2 + \frac{2}{r}\partial_r)\varphi(r)$ where $T\varphi(r) = r\varphi(r)$. Hence or otherwise show that the general radial solution of the wave equation on $I\!R^3$ is of the form
$$u(X, t) = |X|^{-1}(\varphi(|X| + t) + \varphi(|X| - t))$$
for some φ and ψ.

4. Show that for $s > \frac{n}{2}$, $H^s(I\!R^n)$ is an algebra i.e $f, g \in H^s(I\!R^n)$ implies $fg \in H^s(I\!R^n)$ and $||fg||_s \leq C||f||_s||g||_s$. Is this true when $s = 0$?

5. Show that $P(D) = (\frac{\partial}{\partial X_1^2} - \sum\limits_{j=2}^{n} \frac{\partial^2}{\partial X_j^2})$ does not satisfy Hormander's condition i.e $|P^\alpha(\xi)| \leq C_\alpha|\xi|^{-\delta|\alpha|}|P(\xi)|, |\xi|$ large for any $\delta > 0$ whereas $Q(D) = \frac{\partial}{\partial X_1} - \sum\limits_{j=2}^{n} \frac{\partial^2}{\partial X_j^2}$ satisfies the condition with $\delta = \frac{1}{2}$.

6. If $f, g \in s(I\!R^n)$ then for any $s \geq 0$ $(1+|\xi|^2)^{s/2}(fg)^\wedge(\xi) = \int K(\xi, \eta)u(\xi - \eta)\ v(\eta)d\eta$ where $u, v \in L^2(I\!R^n)$ and $K(\xi, \eta)$ satisfies
$$\begin{aligned} K(\xi, \eta) &\leq C_s(1 + |\eta|^2)^{-s/2} \quad \text{if } |\xi - \eta| \geq \frac{1}{2}|\xi|, \\ K(\xi, \eta) &\leq C_s(1 + |\eta - \xi|^2)^{-s/2} \text{ if } |\xi - \eta| \geq \frac{1}{2}|\xi|, \end{aligned}$$

7. For each $w \in S^{n-1}$ define R on $s(I\!R^n)$ by $Rf(s, w) = \int\limits_{u \in w^\perp} f(sw + u)du$. find $Rf(s, w)$ explicitly when $f(X) = e^{-\frac{1}{2}|X|^2}$.

8. Show that the operator T_w defined by $T_w f(s) = \int\limits_{-\infty}^{\infty} e^{i\lambda s}\hat{f}(\lambda w)d\lambda$ takes nonnegative functions into nonnegative functions.

9. If $s > \frac{n}{2}$ and $u, v \in L^2(I\!R^n)$ show that the function w defined by

$$w(\xi) = \int C1 + |\eta|^2)^{-s/2} u(\xi - \eta) v(\eta) d\eta$$

belongs to $L^2(I\!R^n)$ and $||w||_0 leqC ||u||_d ||v||_0$.

10. If f is a C^2 function define u by $u(x, r) = \int\limits_{|y|=1} f(x+ry)\phi\sigma(y)$ for $r > 0$ where $\phi\sigma$ is the normalised surface measure on S^{n-1}. Find a differential equation on $I\!R^n \times (0, \infty)$ satisfied by $u(x1^r)$. Hence or otherwise show that any C^2 function f satisfying

$$\int\limits_{|y|=1} f(X + ry)\phi\sigma(y) = f(X)$$

for all $r > 0$ is actually C^∞.

11. Using the operator T is problem 3 find all radial harmonic functions on $I\!R^3$.

12. Suppose $u_j(y, t),\ j = 1, 2, \ldots n$ are solutions of the one dimensional heat heat equation. Show that $u(X, t) = \prod\limits_{j=1}^{n} u_j(X_j, t)$ is a solution of $(\partial_t - \Delta)u = 0$ on $I\!R^n$. Is the same true for the Laplacian? Why?